Generated on 2025-12-02

Xia's Blog 2025

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2025-12-2

AnyRouter 返回 502

Claude Code

1.1 AnyRouter 返回 502

AnyRouter 白嫖 Claude，偶尔会遇到 502 错误。 参见这个 github comment

首先 Clash 要设置为 TUN 模式，然后删除

```
rm -rf ~/.claude/statsig/*
```

按照其建议设置环境变量

```
echo 'export DISABLE_TELEMETRY=1' >> ~/.zshrc
source ~/.zshrc
```

暂时解决了！

1.1.1 AI 解释

不能全靠别人一张嘴，也许应该知道一下原理

关掉 TELEMETRY（遥测/数据统计）的核心原因很简单：因为它的连接失败导致了整个程序崩溃。 这里是更深层的技术解释，帮你理解发生了什么：

1. 根本原因：网络环境问题

claude 这个命令行工具集成了 Statsig（一个用于功能开关管理和用户行为统计的第三方服务）。 在国外：工具启动 -> 连接 Statsig 服务器 -> 上报"用户启动了程序" -> 成功 -> 程序继续运行。 在国内：工具启动 -> 连接 Statsig 服务器 -> 被防火墙阻断或连接极其不稳定 -> 返回 520 错误或超时。

2. 为什么会导致程序挂掉？（520 Error）

理论上，统计数据上传失败不应该影响主程序使用。但 Claude CLI 的某些版本在错误处理上写得不够好： 它把连接 Statsig 当作了一个"必须完成"的启动步骤。 当服务器返回 520 (Web Server Returned an Unknown Error) 或者连接被重置时，程序抛出了异常，导致整个工具直接退出。 你看到的 API Error: 520，其实不是 DeepSeek 或 Claude 的 AI 接口坏了，而是后台偷偷上传数据的那个接口挂了。

3. DISABLE_TELEMETRY=1 做了什么？

当你设置这个环境变量为 1 时，相当于给程序下达了"静默模式"指令。程序启动时，检测到这个变量。直接跳过 Statsig 初始化和数据上传的代码块。程序就不会去请求那个连不通的服务器，自然就不会报错，直接进入正常的 AI 对话功能。

4. 形象的比喻

想象你要开一辆车（启动 Claude 工具）： 默认情况：车子设计逻辑是"启动前必须向厂家发送一条短信汇报位置"。如果你的手机没信号（网络不通），短信发不出去，车子的电脑就报错，锁死引擎不让你开。 关掉 Telemetry：相当于剪断了那根发短信的电线，或者告诉车载电脑"别发短信了，直接点火"。车子就不再检查短信是否发送成功，直接就能开了。总结 关掉它有百利而无一害： 修复报错：解决 520/Timeout 导致无法使用的问题。 加快启动：省去了建立网络连接的时间，工具启动更快。 保护隐私：你的使用习惯、报错日志不会被上传到服务器。 功能无损：这完全不影响你调用 DeepSeek 或 Claude 的 AI 核心功能。

2025-11-30

1D tV model

2.1 Jordan Wigner transformation

Definition 2.1.1 (Jordan Wigner Transformation)

$$\begin{aligned} c_j &= \left(\prod_{k<j} -\sigma_k^z\right)\frac{1}{2}\sigma_j^- \\ c_j^\dagger &= \left(\prod_{k<j} -\sigma_k^z\right)\frac{1}{2}\sigma_j^+ \\ n_j = c_j^\dagger c_j &= \frac{1}{2}\left(\sigma_j^z + 1\right) \end{aligned} \tag{1}$$

tV model can be viewed as an equivalent XXZ model.

Definition 2.1.2 (tV model)

$$H = -t\sum_j \left(c_j^\dagger c_{j+1} + h.c.\right) + V\sum_j n_j n_{j+1} \tag{2}$$

2.1.1 Hopping term

$$\begin{aligned} c_j^\dagger c_{j+1} &= \left(\prod_{k<j} -\sigma_k^z\right)\frac{1}{2}\sigma_j^+\left(\prod_{l<j+1} -\sigma_l^z\right)\frac{1}{2}\sigma_{j+1}^- \\ &= -\frac{1}{4}\sigma_j^+\sigma_j^z\sigma_{j+1}^- \\ &= \frac{1}{4}\sigma_j^+\sigma_{j+1}^- \end{aligned} \tag{3}$$

2.1.2 Interaction term

$$\begin{aligned} n_j n_{j+1} &= \frac{1}{4}\left(\sigma_j^z + 1\right)\left(\sigma_{j+1}^z + 1\right) \\ &= \frac{1}{4}\left(\sigma_j^z\sigma_{j+1}^z + \sigma_j^z + \sigma_{j+1}^z + 1\right) \end{aligned} \tag{4}$$

perform summation

$$\sum_j n_j n_{j+1} = \frac{1}{4}\left(\sum_j \sigma_j^z\sigma_{j+1}^z + 2\sum_j \sigma_j^z + L\right) \tag{5}$$

Definition 2.1.3 let

$$S_j^\alpha = \frac{1}{2}\sigma_j^\alpha \tag{6}$$

Thus the mapped system becomes

$$\begin{aligned} H &= -t\sum_j \left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right) + V\sum_j S_j^z S_{j+1}^z + V\sum_j S_j^z \\ &= -2t\sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y\right) + V\sum_j S_j^z S_{j+1}^z + V\sum_j S_j^z \end{aligned} \tag{7}$$

which is typically a XXZ model with magnetic field in z direction.

2025-11-30

Eigenstate thermalization hypothesis

3.1 ETH FETH & FP

3.1.1 Statistics

equilibrium: find const and do Gibbs Statistics

Problem. How to formulate non-equilibrium Statistics in a closed system without external bath?

Aside from the non-equilibrium case, quantum case is subtle enough.

In a quantum phase space, there is no real notion of trajectory.

We would talk about closed system, however without external bath, how can we define equilibrium?

3.1.1.1 Quantum Quench System

$H_0|\psi_0\rangle = 0, H_0 \to H(t)$

$$|\psi(t)\rangle = \exp(-iHt)|\psi_0\rangle \tag{8}$$

Problem. For $t \to \infty$, can this be equivalent to Gibbs ensemble?

such that

$$E = \langle\psi|H|\psi\rangle = \frac{1}{Z}\operatorname{tr}(\exp(-\beta H)H) \tag{9}$$

where $\beta = \beta(E)$ is function of E

> **Remark**
>
> Rabi oscillation without bath coupling would not thermalize.

this can only holds for local observables O, since the whole system is pure. it can only be locally viewed as a thermal ensemble.

> **Remark**
>
> The large fraction of pure state could serve as the bath for small subsystem.

> **Remark**
>
> eigen state should be always thermalized.

3.1.1.2 Expectation value

We assume no accidental degeneracy.

$$\langle A(t)\rangle = \sum_i |c_i|^2 A_{ii} + \sum_{i\neq j} c_i^* c_j \exp\big(i(E_i - E_j)t\big)A_{ij} \tag{10}$$

we hope

$$\lim_{t\to\infty} \langle A(t)\rangle_{\beta \mathrm{E}} = \langle A\rangle_{\mathrm{ME}} \tag{11}$$

for local observables.

$$[A]_\infty = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t \langle A(t)\rangle = \sum_i |c_i|^2 A_{ii} \tag{12}$$

3.1.1.2.1 Random Matrix Theory (RMT)

3.1.1.2.1.1 Wigner Matrix

$$\rho(H) = \prod_{ij} f(H_{ii}) g\big(H_{ij}\big) \tag{13}$$

This is not rotational invariant.

3.1.1.2.1.2 Rotation invariance

$$\rho(H) = \rho\big(U^\dagger H U\big) \tag{14}$$

where U is any unitary matrix.

3.1.1.2.1.3 Intersection of above two

$$\rho(H) = \exp\left(-\frac{1}{2}\operatorname{tr} H^2\right) \tag{15}$$

Can check this is both WM and RI.

Interestingly,

$$\rho(H) = \exp(-\beta \operatorname{tr} H) \tag{16}$$

is also both WM and RI.

3.1.1.2.1.4 Haar Random

$$\left\langle U_{i\alpha} U^\dagger_{\alpha' i'}\right\rangle = \frac{1}{D}\delta_{ii'}\delta_{\alpha\alpha'} \tag{17}$$

I'm too lazy to type higher moments.

3.1.1.3 Back to Expectation value

If H is RMT, we can average A_{ii} in the random matrix ensemble. $\langle A_{ii}\rangle = \sum_\alpha \frac{A_\alpha}{D}$, where A is diagonalized in $A = \sum_\alpha A_\alpha |\alpha\rangle\langle\alpha|$

Thus $\langle A_{ii}\rangle$ is independent of i.

$$[A]_\infty = \sum_i |c_i|^2 A_{ii} = \sum_\alpha \frac{A_\alpha}{D} = \langle A\rangle_{\mathrm{ME}} \tag{18}$$

This is too strong, since the thermal state **forgets** all information from the initial state.

3.1.2 ETH

Definition 3.1.1

$$\begin{aligned} \langle A_{ii} \rangle = A(E_i), \langle A_{ii}^2 \rangle = A^2(E_i) \\ \left\langle |A_{ij}|^2 \right\rangle = e^{-S(E^+)} \, |f_A(E^+, \omega)|^2 \end{aligned} \tag{19}$$

where $E^+ = \frac{E_i + E_j}{2}, \omega = E_i - E_j$, f_A is smooth function of E^+, ω, and $e^{S(E^+)}$ is somehow the dimension of local Hilbert space at energy E^+.

Now that,

$$\langle A(t) \rangle = \sum_i |c_i|^2 A(E_i) + \mathcal{O}\left(D^{-\frac{1}{2}}\right) \tag{20}$$

3.1.3 Full ETH (FETH)

3.1.4 Free Probability (FP)

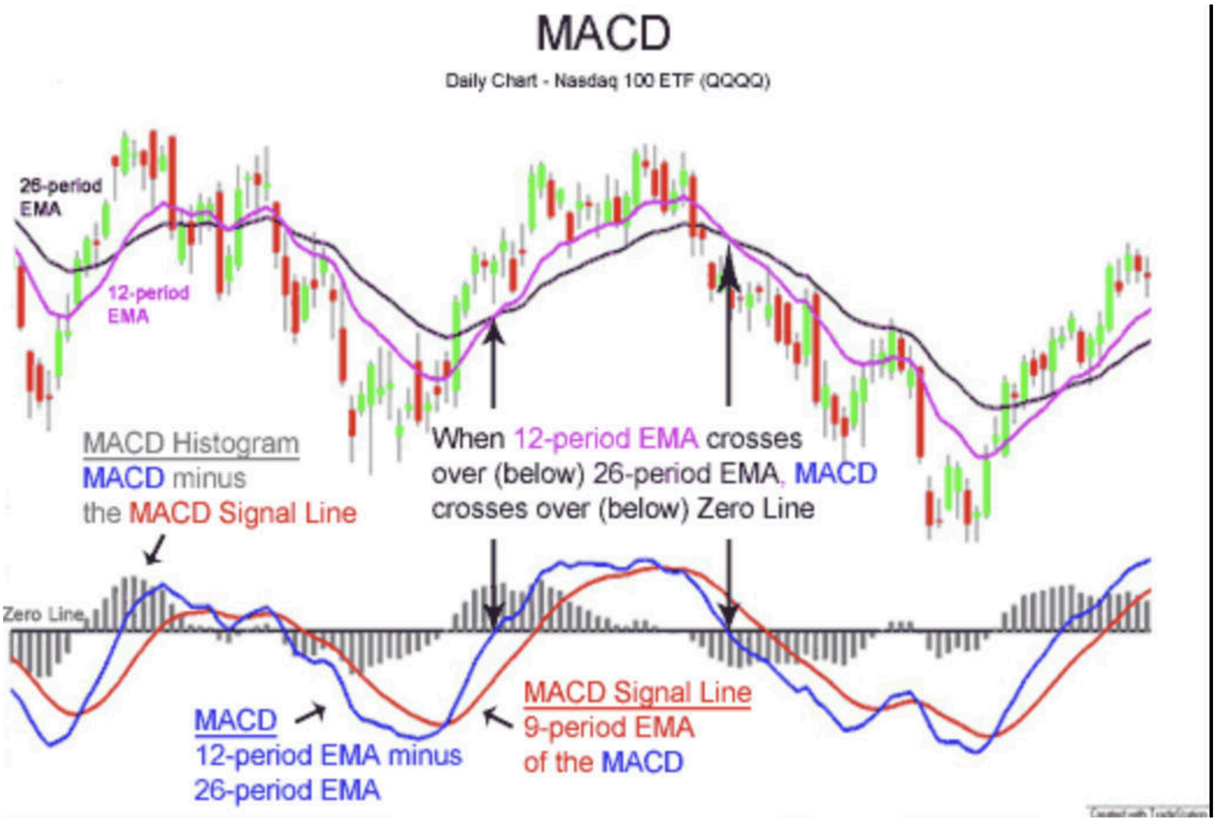
2025-11-29

MACD

Learning Stock

4.1 MACD 指标简介

MACD (Moving Average Convergence Divergence)，中文称为平滑异同移动平均线，用于描述价格趋势的强度，方向，动量和持续时间。



4.1.1 MACD 组成部分

MACD 由三部分组成：

1. DIF 线（快线，白线）：对股价变化反应较快
2. DEA 线（慢线，黄线）：对股价变化反应较慢
3. MACD 柱（红绿柱）：反映 DIF 线和 DEA 线之间的差距

0 轴:中间的分界线。0 轴上方为多头市场（强势），0 轴下方为空头市场（弱势）。

4.1.2 计算方法

略

4.1.3 核心用法

DIF 和 DEA 的区别是判断股价适合短期操作还是长期操作的关键。

4.1.3.1 1. 金叉

- 现象： DIF 线（快线）由下向上穿过 DEA 线（慢线）。
- 含义： 短期动能强于长期动能，股价可能上涨。
- 细节判断:
  - 0 轴下金叉： 通常是反弹信号，不一定是反转，需谨慎。
  - 0 轴上金叉： 属于强势上涨的中继信号，可靠性更高。

4.1.3.2 2. 死叉

- 现象： DIF 线（快线）由上向下穿过 DEA 线（慢线）。
- 含义： 短期动能弱于长期动能，股价可能下跌。
- 细节判断:
  - 0 轴上死叉： 通常是回调信号，不一定是反转，需谨慎。
  - 0 轴下死叉： 属于弱势下跌的中继信号，可靠性更高。

4.1.3.3 3. 0 轴的意义

- 0 轴上方： 多头市场，股价处于强势状态。
- 0 轴下方： 空头市场，股价处于弱势状态。

4.1.3.4 4. 背离 （最具参考价值）

- 现象： 股价走势与 MACD 指标走势出现相反趋势。
- 含义： 可能预示着股价即将反转。
- 细节判断:
  - 多头背离： 股价创新低，但 MACD 未创新低，可能预示股价反弹。
  - 空头背离： 股价创新高，但 MACD 未创新高，可能预示股价回调。

4.1.4 MACD 的优缺点

优点:

1. 稳定性好： 过滤掉了许多短期价格波动的噪音，比起 KDJ 等指标更稳定。
2. 趋势性强： 非常适合用来判断中长期的趋势方向。
3. 背离信号准确： 在捕捉顶部和底部反转时，背离信号参考价值极高。

缺点:

1. 滞后性： 由于是基于移动平均线计算，MACD 对价格变化的反应较慢，可能错过最佳买卖时机。
2. 适用范围有限： 在震荡市中，MACD 可能会频繁发出错误信号，导致亏损。

2025-11-20

Magic in Fermionic System

Reading arXiv:2412.05367v2

5.1 What is Magic?

I would like to put things short. Magic, in the context of quantum computation, is to describe how far a quantum system is from a **stablizer state**, and stablizer state is a state that can be prepared by **Clifford gates**.

However, this definition seems untractable for me to extend to Fermionic systems. As stated in the main paper, *Free Fermionic states(Fermion Gaussian States)* are classically simulatable and thus not magic. If this is the starting point, then it seems to me that the word "magic" is abused.

From now on I would like to state that we are trying to understand the `non-Gaussian-ness` of a Fermionic state

2025-11-16

Many Body Theory

This is a study note of many body theory. Topics will be selected out of my own interest. 主题部分参考（抄写）知乎大黄猫老师的讲义，用英文是因为不用切输入法打字比较快。未经本人允许请勿传播。
It's better to understand than to record.

6.1 Jellium Model

Problem. Jellium model is a 3-dimensional electron gas with a uniform background of positive charge.

The Hamiltonian is given by

$$\begin{aligned} H &= H_{\mathrm{el}} + H_{\mathrm{bg}} + H_{\mathrm{int}} \\ H_{\mathrm{el}} &= \sum_{i=1}^{N} \frac{P_i}{2m} + \frac{e^2}{2} \sum_{i,j} \frac{e^{-\mu|\vec{r}_i-\vec{r}_j|}}{|\vec{r}_i-\vec{r}_j|} \\ H_{\mathrm{bg}} &= \frac{e^2}{2} \int \mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{x}' \frac{n(\vec{x})n(\vec{x}')e^{-\mu|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \\ H_{\mathrm{int}} &= -e^2 \sum_{i=1}^{N} \int \mathrm{d}^3\vec{x} \int \mathrm{d}^3\vec{x} \frac{n(\vec{x})e^{-\mu|\vec{x}-\vec{r}_i|}}{|\vec{x}-\vec{r}_i|} \end{aligned} \tag{21}$$

where we take the column interaction as the Yukawa form

$$V(r) = \frac{e^{-\mu r}}{r} \quad \mu > 0 \tag{22}$$

in order to avoid the divergence of the interaction at the origin. When $\mu = 0$, it becomes the Coulomb interaction.

Assuming that the positive charge density is uniform, and have the same total charge as the electrons, we have

$$n(\vec{x}) = \frac{N}{V} \tag{23}$$

The Yukawa interaction of background positive charge can be integrated out, and we have

$$\begin{aligned} H_{\mathrm{bg}} &= \frac{e^2}{2} \left(\frac{N}{V}\right)^2 \int \mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{x}' \frac{e^{-\mu|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \\ &= \frac{e^2}{2} \left(\frac{N}{V}\right)^2 V \int 4\pi r^2\,\mathrm{d}r \frac{e^{-\mu r}}{r} \\ &= \frac{e^2}{2} \frac{N^2}{V} \frac{4\pi}{\mu^2} \end{aligned} \tag{24}$$

Similarly,

$$H_{\text{int}} = -e^2 \frac{N^2}{V} \frac{4\pi}{\mu^2} \tag{25}$$

where

> **Remark**
>
> $$H_{\text{int}} = -2H_{\text{bg}} \tag{26}$$

The H_{el} term is better written in the second quantization, since it contains the kinetic energy term.

$$H_{\text{el}} = \sum_{k\sigma} \frac{k^2}{2m} c_{k\sigma}^\dagger c_{k\sigma} + \frac{1}{2} \sum_{k,k',q,\sigma,\sigma'} c_{k+q\sigma}^\dagger c_{k'-q\sigma'}^\dagger c_{k'\sigma'} c_{k\sigma} \tag{27}$$

The fourier transform of Yukawa potential is well-known as

$$V(q) = \frac{4\pi}{V} \frac{e^2}{q^2 + \mu^2} \tag{28}$$

The most annoying part is when $q = 0$, the potential diverges. However it is shown to be cancelled by the background energy and the interaction energy. Since

$$H_{\text{el-int}}(q = 0) = \frac{4\pi}{2\mu^2 V} \sum_{k,k',\sigma,\sigma'} c_{k\sigma}^\dagger c_{k'\sigma'}^\dagger c_{k'\sigma'} c_{k\sigma} \tag{29}$$

By the anti-commutation relation, we have

$$\begin{aligned} H_{\text{el-int}}(q = 0) &= -\frac{4\pi e^2}{2\mu^2 V} \sum_{k,k',\sigma,\sigma'} c_{k\sigma}^\dagger c_{k'\sigma'}^\dagger c_{k\sigma} c_{k'\sigma'} \\ &= -\frac{4\pi e^2}{2\mu^2 V} \sum_{k,k',\sigma,\sigma'} c_{k\sigma}^\dagger \left(\delta_{k,k'} \delta_{\sigma,\sigma'} - c_{k\sigma} c_{k'\sigma'}^\dagger\right) c_{k'\sigma'} \\ &= -\frac{4\pi e^2}{2\mu^2 V} (N - N^2) \end{aligned} \tag{30}$$

> **Remark**
>
> In thermodynamics limit, only N^2 term survives, and it cancels the H_{bg} and H_{int} term. This is also why we introduce the background charge density.

Thus Equation 27 is reduced to

Corollary 6.1.0.1

$$H_{\text{el}} = \sum_{k\sigma} \frac{k^2}{2m} c_{k\sigma}^\dagger c_{k\sigma} + \frac{1}{2V} \sum_{k,k',q\neq 0,\sigma,\sigma'} \frac{4\pi e^2}{q^2} c_{k+q\sigma}^\dagger c_{k'-q\sigma'}^\dagger c_{k'\sigma'} c_{k\sigma} \tag{31}$$

We can see the physics by making momentum dimensionless. The typical length is the Bohr radius $a_0 = \frac{1}{me^2}$ (Gaussian unit) . Define $\frac{4}{3}\pi r_0^3 = \frac{V}{N}$, r_0 as the average distance between electrons. Typically, for metals, $r_s = \frac{r_0}{a_0}$ is around 2-6.

Let $\overline{V} = V r_0^{-3}$ and $\overline{k} = k r_0$, then

$$H_{\mathrm{el}} = \frac{e^2}{2 a_0 r_s^2} \left(\sum_{\overline{k}\sigma} \frac{\overline{k}^2}{2m} c_{\overline{k}\sigma}^\dagger c_{\overline{k}\sigma} + \frac{r_s}{\overline{V}} \sum_{\overline{k}, \overline{k}', \overline{q} \neq 0, \sigma, \sigma'} \frac{4\pi}{\overline{q}^2} c_{\overline{k}+\overline{q}\sigma}^\dagger c_{\overline{k}'-\overline{q}\sigma'}^\dagger c_{\overline{k}'\sigma'} c_{\overline{k}\sigma} \right) \tag{32}$$

$\mathrm{Ry} = \frac{e^2}{2a_0}$ is the Rydberg energy as the energy scale of system.

> **Remark**
>
> When $r_s \gg 1$, perturbation theory is unfeasible. We can think that the electron is well-separated. Maybe a Wigner crystal is formed.

6.1.1 Perturbation Calculation: Fock energy

The non-interacting ground state is a Fermi sea $|F\rangle$.

$$E_g^0 = \frac{3}{5} N E_F \tag{33}$$

By pertubation theory, the ground state energy is

$$E_g = E_g^0 + \langle F | H_{\text{el-int}} | F \rangle \tag{34}$$

$H_{\text{el-int}}$ can be decomposed by Wick's theorem.

Theorem 6.1.1 Wick's theorem

$$G(1', 2', 3', ..., n', 1, 2, 3, ..., n) = \sum_P (-1)^P G(P(1'), 1) G(P(2'), 2) ... G(P(n'), n) \tag{35}$$

Then we have

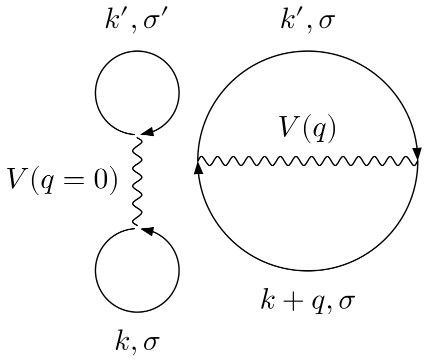
$$\begin{gathered} \langle F | H_{\text{el-int}} | F \rangle = \frac{1}{2} \sum_{k, k', q \neq 0, \sigma, \sigma'} \frac{4\pi e^2}{q^2} \\ \left(l \delta_{q,0} \theta(k_F - |k|) \theta(k_F - |k'|) - \delta_{\sigma,\sigma'} \delta_{k+q,k'} \theta(k_F - |k+q|) \theta(k_F - |k'|) \right) \end{gathered} \tag{36}$$

Definition 6.1.2

The first term is called the direct term or *Hartree term*, and the second term is called the exchange term or *Fock term*.

In our context, Hartree term is cancelled. Only Fock term survives and contributes a negative energy shift.

This two terms can also be drawn in the Feynman diagrams.



Remark

Different from the calculation of scattering amplitude in high energy physics, in condensed matter physics, we are more interested in the *vacuum diagrams*. The above two diagrams are all vacuum diagrams.

We can calculate the Fock term

$$\begin{aligned}\langle F|H_{\text{el-int}}|F\rangle &= -\frac{4\pi e^2 V}{(2\pi)^6}\int \mathrm{d}^3 k\theta(k_F - |k|)\int \mathrm{d}^3 k'\theta(k_F - |k'|)\frac{1}{|k-k'|^2} \\ &= -\frac{4\pi e^2 V}{(2\pi)^6}\int_0^{k_F} 4\pi k^2\,\mathrm{d}k\int_0^{k_F} 2\pi k'^2\,\mathrm{d}k'\int_{-1}^{1}\mathrm{d}\cos\theta\frac{1}{k^2+k'^2-2kk'\cos\theta} \\ &= -\frac{4\pi e^2 V}{(2\pi)^6}\int_0^{k_F} 4\pi k^2\,\mathrm{d}k\int_0^{k_F} 2\pi k'^2\,\mathrm{d}k'\left(-\frac{1}{kk'}\right)\ln\frac{|k-k'|}{|k+k'|} \\ &= \frac{4\pi e^2 V}{(2\pi)^6}\int_0^{k_F} 4\pi k\,\mathrm{d}k\int_0^{k_F} 2\pi k'\,\mathrm{d}k'\ln\frac{|k-k'|}{|k+k'|}\end{aligned} \tag{37}$$

Problem.

$$\int_0^{k_F} 2\pi k'\,\mathrm{d}k'\ln\frac{|k-k'|}{|k+k'|} = ? \tag{38}$$

$$\int_0^{k_F} 2\pi k'\,\mathrm{d}k'\ln\frac{|k-k'|}{|k+k'|} = \int_0^{k} 2\pi k'\,\mathrm{d}k'\ln\frac{k-k'}{k+k'} + \int_k^{k_F} 2\pi k'\,\mathrm{d}k'\ln\frac{k'-k}{k+k'} \tag{39}$$

With the analytical form

$$\int \mathrm{d}x x\ln(x-a) = -\frac{ax}{2} - \frac{x^2}{4} - \frac{1}{2}a^2\ln(a-x) + \frac{1}{2}x^2\ln(-a+x) \tag{40}$$

Write down all terms including the divergent part, hopefully they cancel out.

$$\begin{aligned}
&2\pi\left[\left(kk' - \frac{1}{2}k^2 \ln \frac{k-k'}{k+k'} + \frac{1}{2}k'^2 \ln \frac{k-k'}{k+k'}\right)\Bigg|_0^k + \right. \\
&\left.\left(kk' - \frac{1}{2}k^2 \ln \frac{k'-k}{k+k'} + \frac{1}{2}k'^2 \ln \frac{k'-k}{k+k'}\right)\Bigg|_k^{k_F}\right] \\
&= 2\pi\left[kk_F - \frac{1}{2}k^2 \ln \frac{k_F-k}{k_F+k} + \frac{1}{2}k_F^2 \ln \frac{k-k_F}{k+k_F}\right] \\
&= -2\pi k k_F\left[1 + \frac{k_F^2-k^2}{2k_F k} \ln \frac{|k_F-k|}{|k_F+k|}\right]
\end{aligned} \tag{41}$$

Hence we have

$$\begin{aligned}
\langle F|H_{\text{el-int}}|F\rangle &= -2(2\pi)^2\frac{4\pi e^2 V}{(2\pi)^6}\int_0^{k_F} k^2 k_F \,\mathrm{d}k \left[1 + \frac{k_F^2-k^2}{2k_F k} \ln \frac{|k_F-k|}{|k_F+k|}\right] \\
&= -2(2\pi)^2\frac{4\pi e^2 V}{(2\pi)^6}k_F^4 \int_0^1 x^2 \,\mathrm{d}x \left(1 + \frac{1-x^2}{2x} \ln \frac{|1-x|}{|1+x|}\right)
\end{aligned} \tag{42}$$

where

$$\int_0^1 x^2 \,\mathrm{d}x \left(1 + \frac{1-x^2}{2x} \ln \frac{|1-x|}{|1+x|}\right) = \frac{1}{2} \tag{43}$$

Thus

$$E_{\text{Fock}} = -2(2\pi)^2\frac{4\pi e^2 V}{(2\pi)^6}k_F^4 \frac{1}{2} = -\frac{3e^2}{4\pi}N k_F = -0.916\frac{e^2}{2a_0}\frac{N}{r_s} \tag{44}$$

Which means the energy of 3d jellium electron gas is

Corollary 6.1.2.1

$$E = \frac{E_g}{N} = \frac{e^2}{2a_0}\left(\frac{2.21}{r_s^2} - \frac{0.916}{r_s}\right) \tag{45}$$

Note that with the assumption $r_s \ll 1$, the perturbation is valid.

Remark

The calculation of jellium model gives us a good approximation approach – if your system is not too far from the jellium model, you can ignore the Hartree self energy term. This concept leads to the RPA (Random Phase Approximation) method.

6.1.2 Wigner Crystal

Definition 6.1.3 Wigner crystal

If electron density is *less* than a critical value, the jellium model electron gas will crystallize into a **Wigner crystal**.

Since I'm more familiar with Mott insulator, it seems that the Wigner crystal has much similarity with Mott insulator. They both form when the potential energy dominates the kinetic energy and have strong localization of electrons. However, their context differs. While Wigner crystal forms in continuous space, Mott insulator lives on a lattice system. The Mott physics only have short range interaction, while the Wigner crystal has long range interaction (in Hartree-Fock level, we didn't cut off the long range interaction).

Problem. What is the difference between the Wigner crystal and the Mott insulator?

I cannot currently have a good picture of the Wigner crystal. In Mott insulator with short range interaction, if we perturb one electron (e.g. slightly shift its position), only the nearest neighbor electrons feel that and thus the perturbation is *local* and *screened*.

With some kind help from zhihu, now I would think Wigner crystal as isolated oscillating electrons, while Mott insulator is a system electrons still having strong correlations. Part of the reason is Mott insulator may still have the spin degree of freedom, while we would not say a Wigner crystal is "ferromagnetic" or something.

However, with long rang interaction, I would possibly expect many local minima around the Wigner crystal. This sense comes from the experience of thinking the *Thompson Problem*.

Definition 6.1.4 Thompson Problem

The Thompson problem is a problem of finding the minimum energy configuration of electrons on a sphere.

As far as I know, few configurations are known to have determined lowest energy.

Problem. Is Wigner crystal stable?

If Wigner crystal has many local minima, it is unlikely to survive disorder and quantum fluctuation. Also, the discussion of transportation is hard, since the configuration will vary with time.

A friend has told me since it's quite easy to compute the Wigner crystal by numerics, it's unlikely to have many local minima. Maybe return to this question if some day I have the chance to calculate it.

However, if the Wigner crystal is stable, it is a good candidate for the metal-insulator transition.

6.1.3 CDW and SDW

If the electron density can be written as

$$\begin{aligned} n_\uparrow(\vec{r}) &= \frac{n}{2} + A\cos\left(\vec{Q}\cdot\vec{r}\right) \\ n_\downarrow(\vec{r}) &= \frac{n}{2} + A\cos\left(\vec{Q}\cdot\vec{r} + \varphi\right) \end{aligned} \tag{46}$$

Then the system can form a ground state different from the magnetic etc. ones.

Definition 6.1.5 Charge Density Wave

if $\varphi = 0$, the spin density $S^{z(\vec{r})} = n_\uparrow(\vec{r}) - n_\downarrow(\vec{r}) = 0$, the charge density $n(\vec{r}) = n_\uparrow(\vec{r}) + n_\downarrow(\vec{r}) = n + 2A\cos\left(\vec{Q}\cdot\vec{r}\right)$ is periodic with wave vector $\vec{Q}$. We call this the **Charge Density Wave**.

Spin Density Wave

If $\varphi = \pi$, the spin density $S^{z(\vec{r})} = n_\uparrow(\vec{r}) - n_\downarrow(\vec{r}) = 2A\cos\left(\vec{Q} \cdot \vec{r}\right)$ is periodic with wave vector $\vec{Q}$, the charge density $n(\vec{r}) = n_\uparrow(\vec{r}) + n_\downarrow(\vec{r}) = n$ is uniform. We call this the **Spin Density Wave**.

CDW breaks the translational symmetry, while SDW breaks the spin rotational symmetry and translational symmetry.

Experimentally, CDW can be observed by X-ray diffraction, while SDW can be observed by neutron diffraction.

6.1.4 Hartree-Fock Approximation (Mean Field Theory)

Definition 6.1.6 Hartree-Fock Approximation (Mean Field Theory)

If the operator has non-zero expectation value, we can assume the fluctuation is small and decompose the two operators into the mean field and the fluctuation.

$$AB \approx \langle A\rangle B + \langle B\rangle A - \langle A\rangle\langle B\rangle \tag{47}$$

Revisit Equation 31, we can decouple the 4 operator terms into 2 terms. Possible decomposition is

$$\left\langle c^\dagger_{k+q\sigma} c^\dagger_{k'-q\sigma'} \right\rangle \tag{48}$$

$$\left\langle c^\dagger_{k+q\sigma} c_{k'\sigma'} \right\rangle \tag{49}$$

$$\left\langle c^\dagger_{k+q\sigma} c_{k\sigma} \right\rangle \tag{50}$$

Equation 48 is actually the electron pairing term. This mean field approximation is called the Hartree-Fock-Bogoliubov Approximation. Equation 50 and Equation 49 are the Hartree term and the Fock term we have discussed before.

Remark

Key Observation: By this Hamiltonian decomposition, we can achieve the same results from the perturbation theory

Problem. Why the Hartree-Fock-Bogoliubov Approximation can't be achieved by the perturbation theory?

That's because we use the Fermi sea as the reference state. The Fermi sea without instability can't excite the electron pairing term.

By Fock term, the interaction is now quadratic, and we can solve it exactly.

$$H_{\text{el-int}} = -\frac{1}{V} \sum_{k,k',\sigma} \left(\frac{4\pi e^2}{|k-k'|^2} \left\langle c^\dagger_{k'\sigma} c_{k'\sigma} \right\rangle \right) c^\dagger_{k\sigma} c_{k\sigma} \tag{51}$$

Thus the Hamiltonian of electron gas is now

$$H_{\text{el}} = \sum_k \left(\frac{k^2}{2m} - \frac{1}{V} \sum_{k'} \frac{4\pi e^2}{|k-k'|^2} \left\langle c^\dagger_{k'\sigma} c_{k'\sigma} \right\rangle \right) c^\dagger_{k\sigma} c_{k\sigma} \tag{52}$$

Thus the single electron energy is

$$\varepsilon(k) = \frac{k^2}{2m} - \frac{2e^2 k_F}{\pi} F\left(\frac{k}{k_F}\right) \tag{53}$$

where $F(x) = \frac{1}{2} + \frac{1-x^2}{4x} \ln\left|\frac{1+x}{1-x}\right|$. This is consistent with the result from perturbation theory Equation 44.

The velocity is

$$v = \left.\frac{\partial \varepsilon(k)}{\partial k}\right|_{k=k_F} = \frac{k_F}{m} - \frac{2e^2}{\pi} F'(1) \tag{54}$$

But $F'(1)$ diverges, leading to the un-physical result of infinite Fermi velocity.

> **Remark**
>
> The Fermi velocity tells us how electrons near Fermi surface move. The infinite Fermi velocity is due to the unscreened long range Coulomb interaction.

6.1.5 Thomas-Fermi Approximation

Thomas-Fermi Approximation is a semi-classical one. It considers high density ($r_s \ll 1$) limit, thus the electron density is classical $n(\vec{x})$. The key difficulty in solving the jellium model is the kinetic energy term. It is because of the existing of kinetic energy term that we need to do all things in momentum space. If we can treat the kinetic energy term as a functional of $n(\vec{x})$, we can solve the problem in real space.

The Thomas-Fermi Approximation states, we can regard the kinetic energy term as the strong degenerate non-interacting electron gas, where

$$T_{\mathrm{TF}}[n(\vec{x})] = \int \mathrm{d}^3\vec{x} \frac{3(3\pi^2)^{\frac{2}{3}}}{10m} n(\vec{x})^{\frac{5}{3}} \tag{55}$$

Thus

$$E_{\mathrm{TF}}[n(\boldsymbol{x})] = C(n(\boldsymbol{x}))^{\frac{5}{3}} - Ze^2 \int \mathrm{d}^3\boldsymbol{x} \frac{1}{|\boldsymbol{x}|} n(\boldsymbol{x}) + \frac{e^2}{2} \int \mathrm{d}^3\boldsymbol{x}\, \mathrm{d}^3\boldsymbol{x}' \frac{n(\boldsymbol{x})n(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{56}$$

with the constraint of the total number of electrons, add a Lagrange multiplier.

$$L[n(\boldsymbol{x})] = E_{\mathrm{TF}}[n(\boldsymbol{x})] = \mu\left(\int \mathrm{d}^3\boldsymbol{x} n(\boldsymbol{x}) - N\right) \tag{57}$$

$$\frac{\partial L[n(\boldsymbol{x})]}{\partial n(\boldsymbol{x})} = 0 = Cn(\boldsymbol{x})^{\frac{2}{3}} - Z\frac{e^2}{|\boldsymbol{x}|} + e^2 \int \mathrm{d}^3\boldsymbol{x}' \frac{n(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} - \mu \tag{58}$$

Definition 6.1.7 The effective potential is

$$\varphi(\boldsymbol{x}) = Cn(\boldsymbol{x})^{\frac{2}{3}} \tag{59}$$

$$\varphi(\boldsymbol{x}) = Z\frac{e^2}{|\boldsymbol{x}|} - e^2 \int \mathrm{d}^3\boldsymbol{x}' \frac{n(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} + \mu \tag{60}$$

Take divergence on both sides, we have

$$\nabla^2\varphi(\boldsymbol{x}) = 4\pi e^2(n(\boldsymbol{x}) - Z\delta(\boldsymbol{x})) \tag{61}$$

Thus we have the Thomas-Fermi equation

Definition 6.1.8 The Thomas-Fermi equation is

$$\nabla^2\varphi(\boldsymbol{x}) = 4\pi e^2\left(\left(\frac{\varphi(\boldsymbol{x})}{C}\right)^{\frac{3}{2}} - Z\delta(\boldsymbol{x})\right) \tag{62}$$

6.1.6 Screen Long Range Coulomb Interaction

Thomas Fermi Approximation tells us that the interacting system at high density can be treated as a non-interacting system with an effective potential. This effective potential will affect the density of electron gas at first order. Generally, we can iteratively apply the perturbation until the density converges. There's also another approach from linear response theory to treat it.

Decompose the density into summation of occupation in $\boldsymbol{k}$ space.

$$\rho(\vec{x}) = -ef(\varepsilon_{\boldsymbol{k}}) \tag{63}$$

where $\varepsilon_{\boldsymbol{k}} = \frac{\hbar^2 k^2}{2m} - e\phi(\boldsymbol{x})$

Define

$$n_0(\mu) = \frac{1}{V}\sum_{\boldsymbol{k}}\left(\exp\left(\frac{\hbar^2 k^2}{2m} - \mu\right) + 1\right)^{-1} \tag{64}$$

The density is now

$$\rho(\boldsymbol{x}) = -en_0(\mu + e\phi(\boldsymbol{x})) \tag{65}$$

The "induced" density (or say perturbated) is

$$\begin{aligned}\rho^{\text{induced}}(\boldsymbol{x}) &= -e[n_0(\mu + e\phi(\boldsymbol{x})) - n_0(\mu)] \\ &= -e^2\frac{\partial n_0}{\partial \mu}\phi(\boldsymbol{x})\end{aligned} \tag{66}$$

Remember the linear response theory in real space,

$$\phi^{\text{ext}}(\boldsymbol{x}) = \int \mathrm{d}x'\varepsilon(\boldsymbol{x} - \boldsymbol{x}')\rho(\boldsymbol{x}') \tag{67}$$

Do the Fourier transform,

$$\phi^{\text{ext}}(\boldsymbol{q}) = \varepsilon(\boldsymbol{q})\rho(\boldsymbol{q}) \tag{68}$$

Generally we have

$$\rho^{\text{induced}}(\boldsymbol{q}) = \chi_{c(\boldsymbol{q})}\phi(\boldsymbol{q}) \tag{69}$$

Fourier transform the Possion equation,

$$q^2(\phi(\boldsymbol{q}) - \phi^{\text{ext}}(\boldsymbol{q})) = 4\pi e^2\rho^{\text{induced}}(\boldsymbol{q}) \tag{70}$$

Thus we have

Corollary 6.1.8.1

$$\varepsilon(\boldsymbol{q}) = 1 - \frac{4\pi}{q^2}\chi_c(\boldsymbol{q}) \tag{71}$$

This can be directly achieved by the linear response theory.

Back to Equation 66, we have

$$\chi_{c(\boldsymbol{q})} = -e^2 \frac{\partial n_0}{\partial \mu} \tag{72}$$

which leads to a non-trivial permissivity.

$$\varepsilon(\boldsymbol{q}) = 1 - \frac{4\pi}{q^2}\chi_{c(\boldsymbol{q})} = 1 + \frac{4\pi e^2}{q^2}\frac{\partial n_0}{\partial \mu} \tag{73}$$

Definition 6.1.9 Thomas-Fermi wave vector is

$$k_{\mathrm{TF}}^2 = \frac{4\pi e^2}{q^2}\frac{\partial n_0}{\partial \mu} \tag{74}$$

This non-trivial permissivity leads to the screening of long range Coulomb interaction.

$$\phi(\boldsymbol{q}) = \phi^{\mathrm{ext}} \frac{\boldsymbol{q}}{\varepsilon(\boldsymbol{q})} = \frac{4\pi Q}{q^2 + k_{\mathrm{TF}}^2} \tag{75}$$

This is the Yukawa potential, which screens the long range Coulomb interaction. Because Thomas-Fermi wave vector is approximately same order of k_F. So the typical screening length is the same order of distance between electrons.

> **Remark**
>
> The Thomas-Fermi wave vector is proportional to the DOS at Fermi surface. If we have large DOS at Fermi surface (e.g. flat band), the screening length is short. If the DOS is small, or even zero like at the Dirac point, the screening length is long.

6.1.7 Lindhard Function (RPA)

We can treat the $-e\phi(\boldsymbol{x})$ as an perturbation

The wave function $|k\rangle = \frac{1}{\sqrt{V}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ after first order perturbation is

$$|k^1\rangle = |k\rangle + \sum_{k'} \frac{\langle k'|-e\phi(\boldsymbol{x})|k\rangle}{\varepsilon_{\boldsymbol{k}} - \varepsilon_{\boldsymbol{k}'}} |k'\rangle \tag{76}$$

where $\langle k'|-e\phi(\boldsymbol{x})|k\rangle = -\frac{e}{V}\phi(\boldsymbol{k} - \boldsymbol{k}')$

The induced density is

$$\rho^{\mathrm{induced}}(\boldsymbol{x}) = -\frac{e^2}{V}\sum_{\boldsymbol{k},\boldsymbol{k}'} f(\varepsilon_{\boldsymbol{k}}) \frac{e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}}}{\varepsilon_{\boldsymbol{k}} - \varepsilon_{\boldsymbol{k}'}} \phi(\boldsymbol{k} - \boldsymbol{k}') \tag{77}$$

6.2 QFT at $T = 0$

I will try to follow AGD in this chapter, see how far I can go!

6.2.1 Interaction picture

If we can divide the Hamiltonian as

$$H = H_0 + H_{\text{int}} \tag{78}$$

where H_0 is some Hamiltonian that is relevantly easier to deal with (e.g. the free part), H_{int} contains all the interaction part.

Definition 6.2.1 unofficial definition

Interaction picture says that the state $|\psi\rangle$ evolves with H_{int} and the operator evolves with H_0.

Under interaction picture we have

$$i\frac{\partial}{\partial t}|\psi_I(t)\rangle = H_{\text{int}}(t)|\psi_I(t)\rangle \tag{79}$$

$$\frac{\partial O(t)}{\partial t} = \frac{1}{i}[O(t), H_0] \tag{80}$$

The schrodinger equation under interaction picture Equation 79 can be solved perturbatively. From now on we emit the I subscript for simplicity.

We write out $|\psi(t)\rangle$ as a series:

$$|\psi(t)\rangle = |\psi^0(t)\rangle + |\psi^1(t)\rangle + |\psi^2(t)\rangle + \dots \tag{81}$$

Suppose we know some $|\psi(t_0)\rangle$ at $t = t_0$, in zeroth order where $H_{\text{int}} = 0$, we have

$$|\psi^0(t_0)\rangle = |\psi(t_0)\rangle \tag{82}$$

The first order perturbation is

$$|\psi^1(t_0)\rangle = -i\int_{t_0}^{t} \mathrm{d}t'\, H_{\text{int}}(t')|\psi^0(t_0)\rangle \tag{83}$$

Similarly, the second order perturbation is

$$|\psi^2(t_0)\rangle = -i\int_{t_0}^{t} \mathrm{d}t'\, H_{\text{int}}(t')|\psi^1(t_0)\rangle = -\int_{t_0}^{t} \mathrm{d}t_1 H_{\text{int}}(t_1)\int_{t_0}^{t_1} \mathrm{d}t_2 H_{\text{int}}(t_2)|\psi(t_0)\rangle \tag{84}$$

$$|\psi^n(t_0)\rangle = -i\int_{t_0}^{t} \mathrm{d}t'\, H_{\text{int}}(t')|\psi^{n-1}(t_0)\rangle \tag{85}$$

Definition 6.2.2 The series is called the Dyson series.

Rewrite Equation 81 as

$$|\psi(t)\rangle = S(t, t_0)|\psi(t_0)\rangle \tag{86}$$

where

$$\begin{aligned} S(t,t_0) &= 1 - i\int_{t_0}^{t} \mathrm{d}t_1 H_{\text{int}}(t_1) + \dots + \frac{(-i)^n}{n!}\int_{t_0}^{t}\mathrm{d}t_1 \dots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n H_{\text{int}}(t_1)\dots H_{\text{int}}(t_n) \\ &= 1 - i\int_{t_0}^{t} \mathrm{d}t_1 H_{\text{int}}(t_1) + \dots + \int_{t>t_1>\dots>t_n>t_0} H_{\text{int}}(t_1)\dots H_{\text{int}}(t_n)\,\mathrm{d}t_1 \dots \mathrm{d}t_n \end{aligned} \tag{87}$$

Consider the case that time order $t > t_1 > \dots > t_n > t_0$ doesn't hold. We insert an operator T which is the operation to keep the time order. Thus the upper bound of each integral can be changed to t with a price of $n!$ duplicate terms.

$$S^{(n)}(t,t_0) = \frac{(-i)^n}{n!}\int_{t_0}^{t}\mathrm{d}t_1 \dots \int_{t_0}^{t}\mathrm{d}t_n T[H_{\text{int}}(t_1)\dots H_{\text{int}}(t_n)] \tag{88}$$

That the time evolution operator $S(t,t_0)$ is the time-ordered exponential of $H_{\text{int}}(t)$,

Definition 6.2.3

$$S(t,t_0) = T\exp\left(-i\int_{t_0}^{t}\mathrm{d}t' H_{\text{int}}(t')\right) \tag{89}$$

Note the Time-ordering operator behaves differently in Fermionic and Bosonic systems.

Corollary 6.2.3.1 In Bosonic system

$$T\left[O_1(t_1)O_2(t_2)\dots O_{N(t_N)}\right] = O_{P_1}\left(t_{P_1}\right)O_{P_2}\left(t_{P_2}\right)\dots O_{P_N}\left(t_{P_N}\right) \tag{90}$$

While Fermionic system there's an extra sign

$$T\left[F_1(t_1)F_2(t_2)\dots F_{N(t_N)}\right] = (-1)^P F_{P_1}\left(t_{P_1}\right)F_{P_2}\left(t_{P_2}\right)\dots F_{P_N}\left(t_{P_N}\right) \tag{91}$$

6.3 A Glimpse into Topology

6.3.1 Why is Landau Level State an Ideal Topological Flat Band?

Ref. 空扬笔记

Definition 6.3.1 The Hamiltonian of a free electron in a magnetic field is

$$H = \frac{1}{2M}\left(\left(\hat{\vec{p}} - \vec{A}(\boldsymbol{r})\right)^2\right) + V(r) \tag{92}$$

where $\hbar = e = 1$ and $V(r) = V(r+R)$ is periodical.

This Hamiltonian doesn't commute with the translation operator $\mathcal{T}$, while it commutes with a **magnetic translation operator** $\mathcal{M}(\boldsymbol{R})$

$$\mathcal{M}(\boldsymbol{R},\boldsymbol{r}) \equiv e^{-i\xi_{\boldsymbol{R}}(\boldsymbol{r})}\mathcal{T}(\boldsymbol{R}) \tag{93}$$

where $\xi_{\boldsymbol{R}}(\boldsymbol{r})$ is defined as

$$\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{R}) = \boldsymbol{A}(\boldsymbol{r}) + \nabla\xi_{\boldsymbol{R}}(\boldsymbol{r}) \tag{94}$$

Check that!

$$[\mathcal{T}(\boldsymbol{R}), \boldsymbol{A}^2(\boldsymbol{r})] = \boldsymbol{A}(\boldsymbol{r})\mathcal{T}(\boldsymbol{R})\boldsymbol{A}(\boldsymbol{r}) - \boldsymbol{A}^2(\boldsymbol{r})\mathcal{T}(\boldsymbol{R}) + \boldsymbol{A}(\boldsymbol{r}+\boldsymbol{R})\cdot\boldsymbol{A}(\boldsymbol{r}) = 2\boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{R}) \tag{95}$$

$$\begin{aligned}[\mathcal{M}(\boldsymbol{R},\boldsymbol{r}), \boldsymbol{p}^2] &= e^{-i\xi_{\boldsymbol{R}}(\boldsymbol{r})}\boldsymbol{p}^2\mathcal{T}(\boldsymbol{R}) - \boldsymbol{p}\cdot\left(-\nabla\xi_{\boldsymbol{R}}(\boldsymbol{r})e^{-i\xi_{\boldsymbol{R}}(\boldsymbol{r})}\mathcal{T}(\boldsymbol{R})\right) - \boldsymbol{p}\cdot\left(e^{-i\xi_{\boldsymbol{R}}(\boldsymbol{r})}\boldsymbol{p}\mathcal{T}(\boldsymbol{R})\right) \\ &= -i\nabla^2\xi_{\boldsymbol{R}}(\boldsymbol{r})e^{-i\xi_{\boldsymbol{R}}(\boldsymbol{r})}\mathcal{T}(\boldsymbol{R}) - (\nabla\xi_{\boldsymbol{R}}(\boldsymbol{r}))^2 e^{-i\xi_{\boldsymbol{R}}(\boldsymbol{r})}\mathcal{T}(R)\end{aligned} \tag{96}$$

$$\begin{aligned}&[\mathcal{M}(\boldsymbol{R},\boldsymbol{r}), -\boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{r}) - \boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{p} + \boldsymbol{A}^2(\boldsymbol{r})] \\ &= e^{-i\xi_{\boldsymbol{R}}(\boldsymbol{r})}[-\boldsymbol{p}\cdot\mathcal{T}(\boldsymbol{R})\boldsymbol{A}(\boldsymbol{r}) - \mathcal{T}(\boldsymbol{R})\boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{p} + \mathcal{T}(R)\boldsymbol{A}^2(\boldsymbol{r})] \\ &-[-\boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{r})e^{-i\xi_{\boldsymbol{R}}(\boldsymbol{r})}\mathcal{T}(\boldsymbol{R}) - \boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{p}\left(e^{-i\xi_{\boldsymbol{R}}(\boldsymbol{r})}\mathcal{T}(\boldsymbol{R})\right) \\ &+\boldsymbol{A}^2(\boldsymbol{r})e^{-i\xi_{\boldsymbol{R}}(\boldsymbol{r})}\mathcal{T}(\boldsymbol{R})] \\ &= e^{-i\xi_{\boldsymbol{R}}(\boldsymbol{r})}[-\boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{R}) - \nabla\xi_{\boldsymbol{R}}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r})\mathcal{T}(\boldsymbol{R}) - \boldsymbol{A}(r)\cdot\nabla\xi_{\boldsymbol{R}}(\boldsymbol{r})\mathcal{T}(\boldsymbol{R}) \\ &-\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{R})\cdot\boldsymbol{p} + 2\boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{R})]\end{aligned} \tag{97}$$

The two terms cancel.

6.4 BCS theory Is Deep

6.4.1 BdG formalism

The BdG (Bogoliubov-de Genes) Hamiltonian can be derived from the negative U real space Hubbard model

Definition 6.4.1 Attractive Hubbard model

$$H = -t\sum_{\langle ij\rangle,\sigma} c^\dagger_{i\sigma}c_{j\sigma} - U\sum_i c^\dagger_{i\uparrow}c_{i\uparrow}c^\dagger_{i\downarrow}c_{i\downarrow} \tag{98}$$

Using the mean field approxiamation introduced in Equation 47, we have the BdG Hamiltonian,

Definition 6.4.2 Define the order parameter $\Delta_i = -U\langle c_{i\downarrow}c_{i\uparrow}\rangle$, because Cooper pair of Fermions condense at the ground state.

Thus,

$$\begin{aligned}-U\sum_i c^\dagger_{i\uparrow}c_{i\uparrow}c^\dagger_{i\downarrow}c_{i\downarrow} &= -U\sum_i c^\dagger_{i\uparrow}c^\dagger_{i\downarrow}c_{i\downarrow}c_{i\uparrow} \\ &\approx \sum_i\left(\Delta_i c^\dagger_{i\uparrow}c^\dagger_{i\downarrow} + \Delta_i^* c_{i\downarrow}c_{i\uparrow} + \frac{|\Delta_i|^2}{U}\right)\end{aligned} \tag{99}$$

which give the **BdG Hamiltonian**

$$H_{\text{BdG}} = -t\sum_{\langle ij\rangle,\sigma} c^\dagger_{i\sigma}c_{j\sigma} + \sum_i\left(\Delta_i c^\dagger_{i\uparrow}c^\dagger_{i\downarrow} + \Delta_i^* c_{i\downarrow}c_{i\uparrow} + \frac{|\Delta_i|^2}{U}\right) \tag{100}$$

If the system is uniform where $\Delta_i = \Delta$, the Hamiltonian will be diagonalized in Fourier space

$$\begin{aligned}
H_{\mathrm{BdG}} &= \sum_{\langle ij\rangle\sigma} -t\frac{1}{N_s}\sum_{p,q} e^{i(-pr_i+qr_j)} c^\dagger_{p\sigma}c_{q\sigma} + \frac{\Delta}{N_s}\sum_i\sum_{p,q} e^{-i(p+q)r_i} c^\dagger_{p\uparrow}c^\dagger_{q,\downarrow} \\
&+\frac{\Delta^*}{N_s}\sum_i\sum_{p,q} e^{i(p+q)r_i} c_{p\downarrow}c_{q\uparrow} + N_s\frac{|\Delta|^2}{U} \\
&= -t\sum_{k\sigma}\left(\sum_{\delta r} e^{ik\delta r}\right) c^\dagger_{k\sigma}c_{k\sigma} + \sum_k\left(\Delta c^\dagger_{k\uparrow}c_{-k\downarrow} + \Delta^* c_{k\downarrow}c_{-k\uparrow}\right) + N_s\frac{|\Delta|^2}{U} \\
&= \sum_{k\sigma}\xi_k c^\dagger_{k\sigma}c_{k,\sigma} + \sum_k\left(\Delta c^\dagger_{k\uparrow}c_{-k\downarrow} + \Delta^* c_{k\downarrow}c_{-k\uparrow}\right) + N_s\frac{|\Delta|^2}{U}
\end{aligned} \tag{101}$$

The kinetic energy can be rewritten as

$$\sum_k \xi_k c^\dagger_{k\sigma}c_{k\sigma} = \sum_k \xi_k\left(c^\dagger_{k\sigma}c_{k\sigma} - c_{-k\sigma}c^\dagger_{-k\sigma} + 1\right) \tag{102}$$

Definition 6.4.3 Nambu Spinor in k space

$$\psi_k = \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} \tag{103}$$

Thus

$$H_{\mathrm{BdG}} = \psi^\dagger_k\begin{pmatrix} \xi_k & \Delta \\ \Delta^* & -\xi_k \end{pmatrix}\psi_k + N_s\frac{|\Delta|^2}{U} + \sum_k \xi_k \tag{104}$$

The constant part $\sum_k \xi_k$ can be dropped.

Write the 2×2 matrix in the form of Pauli matrix

$$H_{\mathrm{BdG}} = \psi^\dagger_k[\varepsilon_k\tau_3 + \Delta_1\tau_1 + \Delta_2\tau_2]\psi_k + N_s\frac{|\Delta|^2}{U} \tag{105}$$

where

$$\vec{\tau} = (\tau_1,\tau_2,\tau_3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \tag{106}$$

The notation τ is used to distinguish Pauli matrix in charge space from σ in spin space.

Thus

$$H = \sum_k \psi^\dagger_k\left(\vec{h}_k\cdot\vec{\tau}\right)\psi_k + N_s\frac{|\Delta|^2}{U} \tag{107}$$

where

$$\vec{h}_k = (\Delta_1,\Delta_2,\xi_k) \tag{108}$$

acts as a Weiss field $B_k = -h_k$

Remark

The important thing is the Zeeman field here is momentum dependent.

Unlike normal metals, the Weiss field of superconductor remains finite at the Fermi energy, giving rise to a gap in the excitation spectrum.

The energy gap is describe by the Weiss field, where

$$E_k \equiv \left|\vec{B}_k\right| = \sqrt{|\Delta_1|^2 + |\Delta_2|^2 + \xi_k^2} = \text{quasi particle energy} \tag{109}$$

The gap is $2E_k$

This gap is caused by non-zero density of Cooper pair at the ground state.

The Weiss field can be decomposed into magnitude and direction parts.

$$\begin{aligned} \vec{B}_k &= -E_k \hat{n}_k \\ \hat{n}_k &= \left(\frac{\Delta_1}{E_k}, \frac{\Delta_2}{E_k}, \frac{\xi_k}{E_k}\right) \end{aligned} \tag{110}$$

The direction can be described by the θ_k angle,

$$\cos\theta_k = \frac{\xi_k}{E_k} \tag{111}$$

In the ground state the isospin is parallel to the field, which give the minimum energy of $-\vec{B} \cdot \vec{\tau}$.

$$\left\langle \psi_k^\dagger \vec{\tau} \psi_k \right\rangle = -\hat{n}_k = -(\sin\theta_k, 0, \cos\theta_k) \tag{112}$$

Here we choose the phase of Δ, letting $\Delta_2 = 0$

Thus we can represent all variables, with Δ and θ_k,

$$\langle \tau_{3k} \rangle = \left\langle n_{k\uparrow} + n_{-k\downarrow} - 1 \right\rangle = -\cos\theta_k = -\frac{\xi_k}{\sqrt{\xi_k^2 + \Delta^2}} \tag{113}$$

$$\langle \tau_{1k} \rangle = \left\langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + c_{-k\downarrow} c_{k\uparrow} \right\rangle = \frac{\Delta}{\sqrt{\xi_k^2 + \Delta^2}} \tag{114}$$

and $\langle \tau_{2k} \rangle = 0$ tells that $\left\langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \right\rangle = \left\langle c_{-k\downarrow} c_{k\uparrow} \right\rangle = -\frac{1}{2}\sin\theta_k$

Thus the consistent equation tells that

$$\Delta = -U \left\langle c_{i\downarrow} c_{i\uparrow} \right\rangle = -\frac{U}{N_s} \left\langle \sum_{p,q} e^{-i(p+q)r_i} c_{p\downarrow} c_{q\uparrow} \right\rangle \tag{115}$$

In uniform case, $\Delta = -\frac{U}{N_s} \sum_i \left\langle c_{i\downarrow} c_{i\uparrow} \right\rangle = -\frac{U}{N_s} \sum_k \left\langle c_{k\downarrow} c_{-k\uparrow} \right\rangle$

Thus we have the self-consistent equation

Corollary 6.4.3.1 BCS gap equation at $T = 0$

$$\Delta = -\frac{U}{N_s} \int_{|\xi_k|<\omega_D} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\Delta}{2\sqrt{\xi_k^2 + \Delta^2}} \tag{116}$$

6.4.2 BdG Equation in Continuum

Ref. **Bogoliubov-de Gennes Method and Its Applications**

The BdG approach relies on the assumption that there exist well-defined quasi-particles in SC.

> **Remark**
>
> By Ref. BdG is correct in the weak-coupling regime, but also yiels qualitative results in strong-coupling regime.

Definition 6.4.4 The attractive interaction Hamiltonian is

$$H = \int \mathrm{d}\boldsymbol{r} \psi_\alpha^\dagger(\boldsymbol{r}) h_\alpha(\boldsymbol{r}) \psi_\alpha(\boldsymbol{r}) - \frac{1}{2} \int \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{r}' \psi_\alpha^\dagger(\boldsymbol{r}) \psi_\beta^\dagger(\boldsymbol{r}') V_{\text{eff}}(\boldsymbol{r}, \boldsymbol{r}') \psi_\beta(\boldsymbol{r}') \psi_\alpha(\boldsymbol{r}) \tag{117}$$

where $h_\alpha(\boldsymbol{r})$ is the single particle Hamiltonian defined as

$$h_\alpha(\boldsymbol{r}) = \frac{\left[\frac{\hbar}{i} \nabla_{\boldsymbol{r}} + \frac{e}{c} \boldsymbol{A}(\boldsymbol{r})\right]^2}{2m} - e\varphi(\boldsymbol{r}) + \alpha \mu_B H(\boldsymbol{r}) - \mu \tag{118}$$

With the presence of vector potential, $h_\alpha(\boldsymbol{r})^* \neq h_\alpha(\boldsymbol{r})$

Note that in SC state the Hatree-Fock channel mean field can be absorbed into the chemical potential, we will only consider the particle-particle pairing channel which terms **Bogoliubov-Hatree-Fock** mean field approxiamation. In case of s-wave superconductor, we only consider singlet-singlet pairing.

This gives the effective Hamiltonian

$$\begin{aligned} H_{\text{eff}} = &\int \mathrm{d}\boldsymbol{r} \psi_\alpha^\dagger(\boldsymbol{r}) h_\alpha(\boldsymbol{r}) \psi_\alpha(\boldsymbol{r}) + \int \int \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{r}' \left[\Delta(\boldsymbol{r}, \boldsymbol{r}') \psi_\alpha^\dagger(\boldsymbol{r}) \psi_\beta^\dagger(\boldsymbol{r}') + h.c.\right] \\ &+ \int \int \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{r}' \frac{|\Delta(\boldsymbol{r}, \boldsymbol{r}')|^2}{V_{\text{eff}}(\boldsymbol{r}, \boldsymbol{r}')} \end{aligned} \tag{119}$$

where

$$\Delta(\boldsymbol{r}, \boldsymbol{r}') = -V_{\text{eff}}(\boldsymbol{r} - \boldsymbol{r}') \langle \psi_\downarrow(\boldsymbol{r}') \psi_\uparrow(\boldsymbol{r}) \rangle \tag{120}$$

Definition 6.4.5 Bogoliubov canonical transformation is helpful.

$$\psi_\sigma(\boldsymbol{r}) = \sum_n \left[u_{n\sigma}(\boldsymbol{r}) \gamma_n - \sigma v_{n\sigma} \gamma_n^\dagger\right] \tag{121}$$

where γ_n is a Fermionic operator. In lattice system the scope of n is restricted by the lattice sites, while in continuum, n can reach infinity as a field operator.

Suppose the Hamiltonian is diagonalized on the γ_n basis,

$$H_{\text{eff}} = E_0 + \sum_n E_n \gamma_n^\dagger \gamma_n \tag{122}$$

Thus we have,

$$\begin{aligned} \left[\gamma_n^\dagger, H_{\text{eff}}\right] &= -E_n \gamma_n^\dagger \\ \left[\gamma_n, H_{\text{eff}}\right] &= E_n \gamma_n \end{aligned} \tag{123}$$

These are the equations of motion (EOM) for γ_n, compared with the EOM of field operators $\psi(\boldsymbol{r})$, we have equations for u_n and v_n.

$$\begin{aligned} h_\uparrow(\boldsymbol{r}) u_\uparrow^n(\boldsymbol{r}) + \int \mathrm{d}\boldsymbol{r}' \Delta(\boldsymbol{r}, \boldsymbol{r}') v_\downarrow^n(\boldsymbol{r}') &= E_n u_\uparrow^n(\boldsymbol{r}) \\ h_\downarrow(\boldsymbol{r}) u_\downarrow^n(\boldsymbol{r}) + \int \mathrm{d}\boldsymbol{r}' \Delta(\boldsymbol{r}, \boldsymbol{r}') v_\uparrow^n(\boldsymbol{r}') &= E_n u_\downarrow^n(\boldsymbol{r}) \\ \int \mathrm{d}\boldsymbol{r}' \Delta^*(\boldsymbol{r}, \boldsymbol{r}') u_\downarrow^n(\boldsymbol{r}') - h_\uparrow^*(\boldsymbol{r}) v_\uparrow^n(\boldsymbol{r}) &= E_n v_\uparrow^n(\boldsymbol{r}) \\ \int \mathrm{d}\boldsymbol{r}' \Delta^*(\boldsymbol{r}, \boldsymbol{r}') u_\uparrow^n(\boldsymbol{r}') - h_\downarrow^*(\boldsymbol{r}) v_\downarrow^n(\boldsymbol{r}) &= E_n v_\downarrow^n(\boldsymbol{r}) \end{aligned} \tag{124}$$

Problem. The h^* here is a little bit weird.

The 4 equations can be block diagonalized to 2 sets of 2 equations, since only $u_\uparrow^n(\boldsymbol{r})$ and $v_\downarrow^n(\boldsymbol{r}')$ are coupled, so are $u_\downarrow^n(\boldsymbol{r}')$ and $v_\uparrow^n(\boldsymbol{r})$.

> **Remark**
>
> This is such a messy formalism! I would stop here.

6.4.3 Non-uniform BCS theory

6.4.3.1 Anderson Theorem

Ref. 大黄猫笔记

6.4.3.2 Non-uniform BdG

Ref. 大黄猫笔记

When system is non-uniform E.g. disorder and boundary condition kicks in.

Introduce Nambu Spinor with $2N_s$ components

$$\hat{c} = \left(c_{1\uparrow}, c_{2\uparrow}, ..., c_{N_s\uparrow}, c_{1\downarrow}^\dagger, c_{2\downarrow}^\dagger, ..., c_{N_s\downarrow}^\dagger\right) \tag{125}$$

Thus H_{BdG} can be written as

$$H_{\text{BdG}} = \hat{c}^\dagger H \hat{c} + \sum_i \frac{|\Delta_i|^2}{U} \tag{126}$$

Where

$$H = \begin{pmatrix} \hat{H}_0 & \hat{\Delta} \\ \hat{\Delta}^\dagger & -\hat{H}_0 \end{pmatrix} \tag{127}$$

$\hat{\Delta}$ is $N_s \times N_s$ diagonal matrix for order parameter, and H_0 is the tight-binding matrix.

Definition 6.4.6 The impurity Hamiltonian

$$H = \sum_{\langle ij \rangle, \sigma} \left[-t_{ij} - \mu \delta_{ij} \right] c_{i\sigma}^{\dagger} c_{j\sigma} + \sum_{i} \Delta_i c_{i\uparrow}^{\dagger} c_{i\downarrow} + H.c. + \sum_{\sigma} (\epsilon_I + J\sigma) c_{I\sigma}^{\dagger} c_{I\sigma} \tag{128}$$

where ϵ_I and J are the strength of non-magnetic and ferromagnetic impurity scattering.

2025-11-16

Supernematic

Studying Dan Mao's paper: https://arxiv.org/abs/2511.10642

7.1 Preface

I think the math inside is exciting and the states look like the frustrated hopping ground state that I have learnt from Congjun's papers, where the p-orbital ground state of hubbard model at $\frac{1}{6}$ filling, gives exact solvable Wigner crystal state, with a certain "chirality".

However, I believe Congjun's state never breaks rotational symmetry.

7.2 Questions

Instead of following the paper line by line, I will first list some questions that I have in mind when reading the paper.

1. *Big Question*: Is the rotation symmetry spontaneously broken? If so, what's the order parameter? What's the Goldstone mode?
2. How is "cluster-charge" Hamiltonian motivated from the underlying physics? What are the key physical insights that lead to this Hamiltonian?

2025-04-14

SCFQFT

Self-Consistent Quantum Field Theory notes. Ref Haussmann 2003. Thanks Jiaxin and Yutai for discussion and guidance.

8.1 SCF equations

Anomalous Green's function

$$\left(G^{\alpha_1\alpha_2}_{\sigma_1\sigma_2}\right) = \begin{pmatrix} \delta_{\sigma_1\sigma_2} G(\boldsymbol{r},\tau) & \varepsilon_{\sigma_1\sigma_2} F(\boldsymbol{r},\tau) \\ -\varepsilon_{\sigma_1\sigma_2} F^*(-\boldsymbol{r},\tau) & -\delta_{\sigma_1\sigma_2} G(-\boldsymbol{r},-\tau) \end{pmatrix} \tag{129}$$

The spin index σ_1, σ_2 can be omitted, since the 2×2 spin sector can be always block diagonalized.

$$\left(G_{\alpha_1\alpha_2}(\boldsymbol{r},\tau)\right) = \begin{pmatrix} G(\boldsymbol{r},\tau) & F(\boldsymbol{r},\tau) \\ F^*(-\boldsymbol{r},\tau) & -G(-\boldsymbol{r},-\tau) \end{pmatrix} \tag{130}$$

Dyson equation

$$G^{-1}_{\alpha_1\alpha_2}(\boldsymbol{k},\omega_n) = -i\hbar\omega_n\delta_{\alpha_1\alpha_2} + (\varepsilon_{\boldsymbol{k}} - \mu)\gamma_{\alpha_1\alpha_2} - \Sigma_{\alpha_1\alpha_2}(\boldsymbol{k},\omega_n) \tag{131}$$

where

$$\left(\gamma_{\alpha_1\alpha_2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{132}$$

With ladder approximation, the self energy equation is

$$\Sigma_{\alpha_1\alpha_2}(\boldsymbol{r},\tau) = \Sigma^1_{\alpha_1\alpha_2}\delta(\boldsymbol{r})\hbar\delta(\tau) + G_{\alpha_2\alpha_1}(-\boldsymbol{r},-\tau)\Gamma_{\alpha_1\alpha_2}(\boldsymbol{r},\tau) \tag{133}$$

Do Fourier

$$
\begin{aligned}
&\Sigma_{\alpha_1\alpha_2}(\boldsymbol{k},\omega_n) = \\
&\int \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}\tau \exp(-i\boldsymbol{k}\boldsymbol{r}+i\omega_n\tau)\Big(\Sigma^1_{\alpha_1\alpha_2}\delta(\boldsymbol{r})\delta(\tau)+G_{\alpha_2\alpha_1}(-\boldsymbol{r},-\tau)\Gamma_{\alpha_1\alpha_2}(\boldsymbol{r},\tau)\Big) \\
&= \Sigma^1_{\alpha_1\alpha_2} + \\
&\int \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}\tau \exp(-i\boldsymbol{k}\boldsymbol{r}+i\omega_n\tau)\int\frac{\mathrm{d}^3\boldsymbol{k}_1}{(2\pi)^3}\frac{1}{\beta}\sum_{\omega_n^1}\exp(i\boldsymbol{k}_1\boldsymbol{r}-i\omega_n^1\tau)G_{\alpha_2\alpha_1}(\boldsymbol{k}_1,\omega_n^1) \\
&\int\frac{\mathrm{d}^3\boldsymbol{k}_2}{(2\pi)^3}\frac{1}{\beta}\sum_{\omega_n^2}\exp(i\boldsymbol{k}_2\boldsymbol{r}-i\omega_n^2\tau)\Gamma_{\alpha_1\alpha_2}(\boldsymbol{k}_2,\omega_n^2) \\
&= \Sigma^1_{\alpha_1\alpha_2} + \\
&\int\frac{\mathrm{d}^3\boldsymbol{k}_1\,\mathrm{d}^3\boldsymbol{k}_2}{(2\pi)^6}\frac{1}{\beta^2}\sum_{\omega_n^1,\omega_n^2}\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\delta(\omega_n-\omega_n^1-\omega_n^2)G_{\alpha_2\alpha_1}(\boldsymbol{k}_1,\omega_n^1)\Gamma_{\alpha_1\alpha_2}(\boldsymbol{k}_2,\omega_n^2) \\
&= \Sigma^1_{\alpha_1\alpha_2} + \int\frac{\mathrm{d}^3\boldsymbol{k}_1}{(2\pi)^3}\frac{1}{\beta}\sum_{\omega_n^1}G_{\alpha_2\alpha_1}(\boldsymbol{k}_1,\omega_n^1)\Gamma_{\alpha_1\alpha_2}(\boldsymbol{k}-\boldsymbol{k}_1,\omega_n-\omega_n^1)
\end{aligned}
\tag{134}
$$

where

$$
\Sigma^1_{\alpha_1\alpha_2}(\boldsymbol{k},\omega_n) = \begin{pmatrix} 0 & VF(\boldsymbol{r}=0,\tau=0) \\ VF^*(\boldsymbol{r}=0,\tau=0) & 0 \end{pmatrix}
\tag{135}
$$

The non-diagonal part is actually the order parameter in standard BCS. This may be our starting point?

$$
\Sigma^1_{\alpha_1\alpha_2} = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix}
\tag{136}
$$

The Bethe-Salpeter equation

$$
\Gamma^{-1}_{\alpha_1\alpha_2}(\boldsymbol{K},\Omega_n) = T^{-1}\delta_{\alpha_1\alpha_2} + M_{\alpha_1\alpha_2}(\boldsymbol{K},\Omega_n)
\tag{137}
$$

where

$$
\begin{aligned}
&M_{\alpha_1\alpha_2}(\boldsymbol{K},\Omega_n) = \\
&\int\frac{\mathrm{d}^d k}{(2\pi)^d}\left[\frac{1}{\beta}\sum_{\omega_n}G_{\alpha_1\alpha_2}\left(\frac{1}{2}\boldsymbol{K}-\boldsymbol{k},\Omega_n-\omega_n\right)G_{\alpha_1\alpha_2}\left(\frac{1}{2}\boldsymbol{K}+\boldsymbol{k},\omega_n\right)-\frac{m}{\hbar^2 k^2}\delta_{\alpha_1\alpha_2}\right]
\end{aligned}
\tag{138}
$$

is the regularized pair propagator. Note here the $\boldsymbol{K}$ is the total momentum of the pair, and Ω_n is the **Bosonic** Matsubara frequency, $\Omega_n = \frac{2\pi n}{\beta\hbar}$ corresponding to the total energy of the pair.

Denote $\boldsymbol{K} = (K\sin\Theta\cos\Phi, K\sin\Theta\sin\Phi, K\cos\Theta)$, $\boldsymbol{k} = (k\sin\theta\cos\varphi, k\sin\theta\sin\varphi, k\cos\theta)$, then

$$
\begin{aligned}
\left|\frac{\boldsymbol{K}}{2}\pm\boldsymbol{k}\right| &= \sqrt{\frac{1}{4}K^2+k^2\pm\boldsymbol{K}\cdot\boldsymbol{k}} \\
&= \sqrt{\frac{1}{4}K^2+k^2\pm Kk(\sin\Theta\sin\theta\cos(\Phi-\varphi)+\cos\Theta\cos\theta)}
\end{aligned}
\tag{139}
$$

Decompose Equation 138 into spherical coordinates, we get

$$M_{\alpha_1\alpha_2}(\boldsymbol{K},\Omega_n) = \int_0^\Lambda k^2 \frac{\mathrm{d}k}{(2\pi)^3} \int_0^\pi \sin\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi[...] \tag{140}$$

In numerics we need to substitute all $\mathrm{d}k, \mathrm{d}\theta, \mathrm{d}\varphi$ into $\Delta k, \Delta\theta, \Delta\varphi$

> **Remark**
>
> $\Omega_n - \omega_m = \frac{\pi}{\beta\hbar}(2n - (2m+1))$ could be < 0. I would choose to calculate it manually instead of reading from the pre-computed Green function mesh.

in real Space

$$M_{\alpha_1\alpha_2}(\boldsymbol{r},\tau) = \left[G_{\alpha_1\alpha_2}(\boldsymbol{r},\tau)\right]^2 - c\delta_{\alpha_1\alpha_2}\delta(\boldsymbol{r})\hbar\delta(\tau) \tag{141}$$

where c is defined as a constant with ultraviolet momentum cutoff $\Lambda \to \infty$.

$$\begin{aligned} c &= \frac{\Omega_d}{(2\pi)^d}\frac{m}{\hbar^2}\frac{\Lambda^{d-2}}{d-2} \\ \Omega_d &= \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \end{aligned} \tag{142}$$

c goes to infinity when $d > 2, \Lambda \to \infty$.

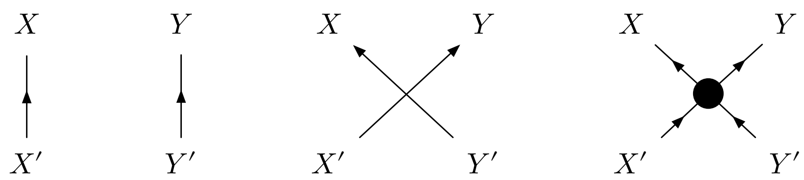
> **Remark**
>
> This UV divergence only occur in continuous system. In lattice model, the momentum cutoff is naturally chosen as $\Lambda = \frac{2\pi}{a}$
>
> If we want to compare with results on the Tianyuan Quantum Simulator, what is the suitable model to use?

Arguably I think continuous model might be better, since the spacing of cold atoms are not as uniform as in the real material.

8.1.1 necessary explanation for SCF equations

The Bethe-Salpeter Equation 137 is introduced since we need to describe the two particle correlations.

$G^2_{XX'YY'} =$



The third term contains a vertex, we define the vertex as $-\Gamma_{UU',WW'}$

The node is painted black, meaning it is resummed by irreducible vertex functions.

If we introduce the pair propagator

$$\chi_{XX',YY'} = \pm G_{XY'}G_{X'Y} \tag{143}$$

which is exactly the second column diagram.

The vertex function can be obtained by

$$\Gamma = \Gamma_1 - \Gamma_1 \chi \Gamma \tag{144}$$

where Γ_1 is the irreducible vertex function. or equivalently

$$\Gamma^{-1} = \Gamma_1^{-1} + \chi \tag{145}$$

This is the Bethe-Salpeter equation.

An approximation we made here is to substitute the irreducible vertex function with

$$\Gamma_1 = \Gamma_0 = s + t + u \tag{146}$$

where s, t, u are the s,t,u channels at tree level.

We will see that Equation 146 leads to a ladder approximation.

The matrix form of pair propagator is

$$\chi_{\alpha_1 \alpha_{1'}, \alpha_2 \alpha_{2'}} = -G_{\alpha_1 \alpha_{2'}}(\boldsymbol{r}_1 - \boldsymbol{r}_2, \tau_1 - \tau_2) G_{\alpha_2 \alpha_{1'}}(\boldsymbol{r}_2 - \boldsymbol{r}_1, \tau_2 - \tau_1) \tag{147}$$

Because of the localization nature of the vertex function, we treat it in the K space, where $\Gamma_0(\boldsymbol{K}, \Omega_n) = \Gamma_0$ is a constant.

Thus

$$\Gamma^{-1}(\boldsymbol{K}, \Omega_n) = \Gamma_0^{-1} + \chi(\boldsymbol{K}, \Omega_n) \tag{148}$$

> **Remark**
>
> These equations include space-time non-homogenous parameters, the number of parameters depends on the discretization of space-time.

> **Remark**
>
> The question is, it will be more clear to discretize in k and ω_n space, more problematic in real space?

8.1.2 mean-field Green's function

Take the weak coupling limit, $a_F \to 0^-$. Because $T^{-1} = \frac{m}{4\pi\hbar^2 a_F}$, in our unit, $T^{-1} = \frac{v}{8\pi\varepsilon_F} k_F$, the leading order of Bethe-Salpeter Equation 137 is

$$\Gamma_{\alpha_1 \alpha_2}(\boldsymbol{K}, \Omega_n) = \frac{4\pi\hbar^2 a_F}{m} \delta_{\alpha_1 \alpha_2} \tag{149}$$

Fourier transform the self-energy Equation 133, we get

$$\begin{aligned}
&\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\beta} \sum_{\omega_n} \exp(i(kr - \omega_n \tau)) \Sigma_{\alpha_1\alpha_2}(\boldsymbol{k}, \omega_n) \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\beta} \sum_{\omega_n} \exp(i(kr - \omega_n \tau)) \Sigma^1_{\alpha_1\alpha_2}(\boldsymbol{k}, \omega_n) \hbar \delta(\boldsymbol{r}) \delta(\tau) \\
&+ \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{\beta^2} \sum_{\omega_n \omega_m} \exp(i((-k_1 + k_2)r) - (-\omega_n + \omega_m)\tau) G_{\alpha_1\alpha_2}(k_1, \omega_n) \Gamma_{\alpha_1\alpha_2}(k_2, \omega_m) \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\beta} \sum_{\omega_n} \exp(i(kr - \omega_n \tau)) \Sigma^1_{\alpha_1\alpha_2}(\boldsymbol{k}, \omega_n) \hbar \delta(\boldsymbol{r}) \delta(\tau) \\
&+ \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \frac{1}{\beta} \sum_{\omega_n} \exp(i((-k_1)r) - (-\omega_n)\tau) G_{\alpha_1\alpha_2}(k_1, \omega_n) T \delta_{\alpha_1\alpha_2} \delta(\boldsymbol{r}) \delta(\tau)
\end{aligned} \tag{150}$$

Thus (because of the orthogonality of the basis functions)

$$\begin{gathered}
\Sigma_{\alpha_1\alpha_2}(\boldsymbol{k}, \omega_n) = \\
\Sigma^1_{\alpha_1\alpha_2}(\boldsymbol{k}, \omega_n) \hbar \delta(\boldsymbol{r}) \delta(\tau) + G_{\alpha_1\alpha_2}(-\boldsymbol{k}, -\omega_n) T \delta_{\alpha_1\alpha_2} \delta(\boldsymbol{r}) \delta(\tau)
\end{gathered} \tag{151}$$

The Fourier transform of self-energy is non-zero only when $r = 0$ and $\tau = 0$.

The $\boldsymbol{r} = 0$, $\tau = 0^-$ component of the normal green function G can be obtained by

$$n_F = -2G(\boldsymbol{r} = 0, \tau = 0^-) \tag{152}$$

Thus

$$\left(\Sigma_{\alpha_1\alpha_2}(\boldsymbol{k}, \omega_n)\right) = \begin{pmatrix} -\frac{2\pi\hbar^2}{m} n_F a_F & \Delta \\ \Delta^* & \frac{2\pi\hbar^2}{m} n_F a_F \end{pmatrix} \tag{153}$$

Insert this into the Dyson Equation 131, we get

$$G^{-1}_{\alpha_1\alpha_2}(\boldsymbol{k}, \omega_n) = \begin{pmatrix} -i\hbar\omega_n + (\varepsilon_{\boldsymbol{k}} - \mu) + \frac{2\pi\hbar^2}{m} n_F a_F & -\Delta \\ -\Delta^* & -i\hbar\omega_n - (\varepsilon_{\boldsymbol{k}} - \mu) - \frac{2\pi\hbar^2}{m} n_F a_F \end{pmatrix} \tag{154}$$

By Fourier transform of Equation 130, we get

$$G_{\alpha_1\alpha_2}(\boldsymbol{k}, \omega_n) = \begin{pmatrix} G(\boldsymbol{k}, \omega_n) & F(\boldsymbol{k}, \omega_n) \\ F^\star(\boldsymbol{k}, -\omega_n) & -G(-\boldsymbol{k}, -\omega_n) \end{pmatrix} \tag{155}$$

> **Remark**
>
> There is a deduction gap in the inversion procedure. Solve it or ask!

You are believed to reproduce this (with a Bogoliubov transformation of the Nambu spinor I think)

$$G(\boldsymbol{k}, \omega_n) = \frac{u_k^2}{-i\hbar\omega_n + E_{\boldsymbol{k}} - \mu} - \frac{v_k^2}{i\hbar\omega_n + E_{\boldsymbol{k}} - \mu} \tag{156}$$

And the anomalous Green's function

$$F(\boldsymbol{k}, \omega_n) = -\frac{\Delta}{|\Delta|} u_k v_k \left[\frac{1}{-i\hbar\omega_n + E_{\boldsymbol{k}} - \mu} + \frac{1}{i\hbar\omega_n + E_{\boldsymbol{k}} - \mu}\right] \tag{157}$$

where

$$\begin{cases} E_{\boldsymbol{k}} = \sqrt{(\bar{\varepsilon}_{\boldsymbol{k}} - \mu)^2 + |\Delta|^2} \\ \bar{\varepsilon}_{\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}} + \frac{2\pi\hbar^2}{m} n_F a_F \\ u_{\boldsymbol{k}}^2 = \frac{1}{2}\left(1 + \frac{\bar{\varepsilon}_{\boldsymbol{k}} - \mu}{E_{\boldsymbol{k}} - \mu}\right) \\ v_{\boldsymbol{k}}^2 = \frac{1}{2}\left(1 - \frac{\bar{\varepsilon}_{\boldsymbol{k}} - \mu}{E_{\boldsymbol{k}} - \mu}\right) \end{cases} \tag{158}$$

Scale the equation of $\bar{\varepsilon}_{\boldsymbol{k}}$ with respect to ε_F and k_F

$$\frac{\bar{\varepsilon}_{\boldsymbol{k}}}{\varepsilon_F} = \frac{\varepsilon_{\boldsymbol{k}}}{\varepsilon_F} + \frac{4}{3\pi}\frac{1}{v} \tag{159}$$

Corollary 8.1.0.1 Equation 156 Equation 157 are mean-field Green's function and anomalous Green's function, at weak coupling limit.

8.1.3 Scale invariance

The SCF equations Equation 131 Equation 133 Equation 137 Equation 138 have 3 parameters, temperature β (hidden in Matsubara summation), the Fermion density n_F (hidden in the chemical potential μ in Dyson Equation 131), the coupling strength a_F^{-1} in Bethe-Salpeter Equation 137.

Because the quadratic energy dispersion $\varepsilon(\boldsymbol{k}) = \frac{\hbar^2 \boldsymbol{k}^2}{2m}$ and the contact potential (in the context of discrete lattice model, i.e. Hubbard model), the interaction term doesn't depend on relative distance, the SCF equations are scale invariant.

$$G_{\alpha_1\alpha_2}(\boldsymbol{k}, \omega_n) = \varepsilon_F^{-1} g_{\alpha_1\alpha_2}\left(\frac{\boldsymbol{k}}{k_F}, \frac{\hbar\omega_n}{\varepsilon_F}\right) \tag{160}$$

$$\Sigma_{\alpha_1\alpha_2}(\boldsymbol{r}, \tau) = \boldsymbol{k}_F^3 \varepsilon_F^2 \sigma_{\alpha_1\alpha_2}\left(k_F \boldsymbol{r}, \frac{\varepsilon_F \tau}{\hbar}\right) \tag{161}$$

The $\Sigma_{\alpha_1\alpha_2}(\boldsymbol{k}, \omega_n)$ is of the dimension of energy, transform to real space and imaginary time gives dimension of $\boldsymbol{k}_F^3 \varepsilon_F$

$$M_{\alpha_1\alpha_2}(\boldsymbol{r}, \tau) = \varepsilon_F^{-2} m_{\alpha_1\alpha_2}\left(k_F \boldsymbol{r}, \frac{\varepsilon_F \tau}{\hbar}\right) \tag{162}$$

$$\Gamma_{\alpha_1\alpha_2}(\boldsymbol{K}, \Omega_n) = \boldsymbol{k}_F^{-3} \varepsilon_F \gamma_{\alpha_1\alpha_2}\left(\frac{\boldsymbol{K}}{k_F}, \frac{\hbar\Omega_n}{\varepsilon_F}\right) \tag{163}$$

Thus we can define dimensionless parameters for SCF equations

Definition 8.1.1 Define dimensionless temperature as

$$\theta = \frac{1}{\beta \varepsilon_F} \tag{164}$$

Define dimensionless coupling strength as

$$v = \frac{1}{k_F a_F} \tag{165}$$

Symbols of temperature and the renormalized interaction strength are abused in the book. From now on I will only use β to represent the temperature.

> **Remark**
>
> However, our SCF equations are derived in the dilute limit where $k_F r_0 \to 0$. As the $k_F r_0$ reaches 1 or be much larger than 1, the Pauli blocking will be extremely relevant.

Tuning the coupling strength, we can see the BCS-BEC crossover.

> **Remark**
>
> **Assumptions used in SCF equations**
> - In Equation 146, the irreducible vertex function Γ_1 is replaced by the bare interaction vertex Γ_0
> - The interaction is assumed local in botsh real space and imaginary time.

These assumptions lead to a ladder approximation.

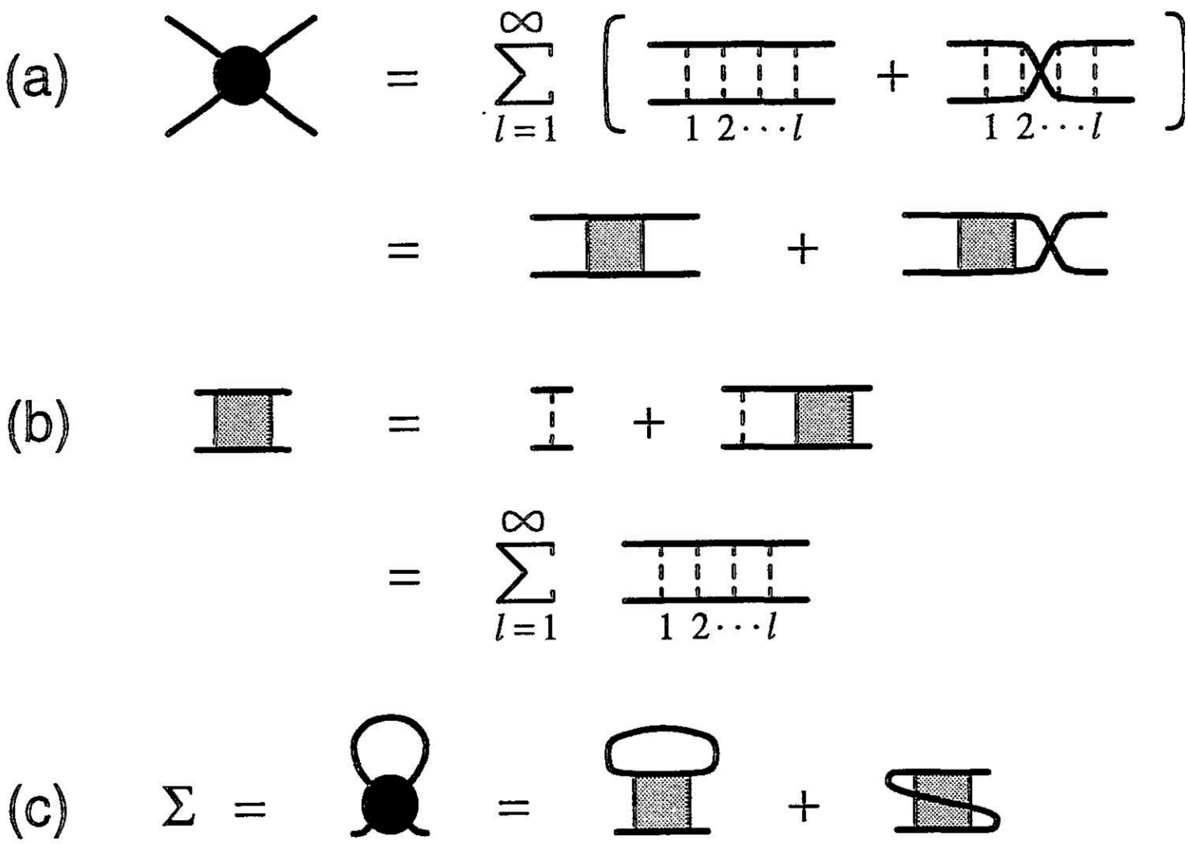


Figure 1: Ladder approximation

In Figure 1 (a) direct and interchange lattice diagrams are considered

> **Remark**
>
> Why don't we have diagrams including multiple interchange vertex?

I think maybe it's because two u channel interaction may go back to the ordinary ladder diagram.

8.1.4 SCF procedure

To study the BCS-BEC crossover region, neither the weak coupling limit nor the strong coupling limit is valid.

We start with inserting the mean-field Green's function Equation 156 Equation 157 into the pair propagator Equation 138, we get (All in the k, ω_n space)

> **Remark**
>
> We used bare vertex to substitute the irreducible vertex. Does it mean the Green function can be also treated freely with this order of approximation?

According to the book, I think the answer is probably NO.

8.1.4.1 Note on Fourier transform

We use the convention

Definition 8.1.2

$$G_{\alpha_1\alpha_2}(\boldsymbol{r}, \tau) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\beta} \sum_{\omega_n} \exp(i(\boldsymbol{k}\boldsymbol{r} - \omega_n \tau)) G_{\alpha_1\alpha_2}(\boldsymbol{k}, \omega_n) \tag{166}$$